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A SECOND-ORDER FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED NONLINEAR DELAY PARABOLIC PROBLEMS WITH PERIODIC INITIAL-BOUNDARY CONDITIONS

Abstract. In this paper, a numerical study for the singularly perturbed nonlinear delay parabolic problems with boundary conditions are made. A finite difference method based on the mesh with adaptive points are proposed. The method employs interpolating quadrature rules containing integral remainder terms with linear basis functions ensuring a second-order accuracy rate on an adaptive mesh. The proposed method exhibits second-order convergence in the space variable, and first-order in the time variable, regardless of the perturbation parameter. Stability analysis is provided, and numerical experiments corroborate the theoretical findings. The results demonstrate the effectiveness of the proposed approach in accurately solving singularly perturbed nonlinear delay parabolic problems with boundary value conditions.

Keywords: time-delayed parabolic-elliptic problems, asymptotic analysis, finite difference method, uniform convergence, Bakhvalov mesh.

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Introduction. In this article, we study the following problem:

$$\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} + b(x,t)u + c(t)u(x,t-r) = f(x,t,u),\ (x,t) \in \mathcal{D}, \tag{1}$$

$$u(x,t) = \psi(x,t),\quad (x,t) \in [0,l] \times [-r,0], \tag{2}$$

$$u(0,t) = u(l,t) = 0,\qquad t \in [0,T] \tag{3}$$

where $0 < \varepsilon < 1$ perturbation parameter, r delay parameter, $T = kr, k \in \mathbb{Z}^+$ $[0,T] = \cup_{i=1}^{k}[(i-1)r, ir]$; $a(x),\ b(x,t), c(t)$ and $f(x,t,u)$ are sufficiently smooth functions. Also, $0 < \alpha \le a(x),\ 0 < b(x,t) \le b^*,\ c^* \ge c(t) > 0$ ve $\left|\frac{\partial f(x,t,u)}{\partial u}\right|,\ \left|\frac{\partial f(x,t,u)}{\partial x}\right| \le M$.

Conditions and methods of research. Such problems arise as mathematical models of the object studied in many fields of science. This problem contains boundary layers where the solution changes rapidly due to its singular perturbation feature. Since the solution changes rapidly at these boundary layers, the algorithm may not be stable due to the derivatives tending to infinity in classical difference problems. For this reason, difference schemes will be established with interpolation quadrature rules containing the integral remainder term and basis functions [1]. Since the coefficient of the highest order term of the differential equation has a small parameter, parameter-dependent singularity occurs. For this reason, choosing the basis functions to zero out the remaining terms in the highest order derivatives is an advantage of the method used. In addition, since the quadrature rules have integral remainder terms, they cause the conditions of the solution function and its derivatives to be alleviated due to the feature of the integral operator. This is another advantage of our method.

In this article, a difference scheme will be established using linear basis functions on a adaptive mesh [6-7].

Research results and discussion. For such problems, you can also refer to the following resources [2-5,13-14].

1. Asymptotic Estimation for Continuous Problem

For the problem (1)-(3), we write:

$$f(x,t,u) = f(x,t,0) + \frac{\partial f(x,t,\tilde{u})}{\partial u}u, \quad \tilde{u} = \gamma u, \quad 0 < \gamma < 1$$

where such that

$$F(x,t) = f(x,t,0), \; B(x,t) = b(x,t) - \frac{\partial f(x,t,\tilde{u})}{\partial u}$$

The problem (1)-(3) changes the following linear problem

$$Lu \equiv \frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} + B(x,t)u + c(t)u(x,t-r) = F(x,t), \tag{4}$$

$$u(x,t) = \psi(x,t), \quad (x,t) \in [0,l]\times[-r,0], \tag{5}$$

$$u(0,t) = u(l,t) = 0, \quad t \in [0,T]\,. \tag{6}$$

where $T = mr$ and $[0,T] = \cup_{p=0}^{m-1} I_p$, $I_p = [pr,(p+1)r]$. Here, we have the domain $[0,l]\times I_0$ for proof. Then will be made the proof for the domain $[0,l]\times I_1$ and using Bellman will be generalized to the $[0,l]\times I_p$ region Bellman.

Lemma 1. The following estimations are true for the solution of the problem (1)-(3):

$$\left\|\frac{\partial^{s+k}u}{\partial x^s \partial t^k}\right\| \le \varepsilon^{-s/2}C_p\left\{\|\psi(x,0)\|^2 + \int_{-r}^{0}\|\psi(x,\tau)\|^2 d\tau + \|F\|_{L_2(D)}^2\right\}, \quad s = 0,1,2\,; k = 0,1; p = 0. \tag{7}$$

Proof. Using the inner product $(Lu,u) = (F,u)$ and energy inequalities, we have the inequality (7).

For $p = 1$, so if we say $u(x,t) = u_1(x,t)$, $(x,t) \in [0,r]$; for $[0,l]\times I_1 = [0,l]\times[r,2r]$; in a similar way we have:

$$\left\|\frac{\partial^{s+k}u}{\partial x^s \partial t^k}\right\| \le \varepsilon^{-\frac{s}{2}} C_p\{\|u_1(x,r)\|^2 + \int_0^r \|u_1(x,\tau)\|^2 d\tau + \|F\|^2_{L_2(D)}\},$$
$s = 0,1,2\,; k = 0,1; p = 1.$

If we continue in this way, we get the following general estimation for

$[0,l] \times I_p = [0,l] \times [pr,(p+1)r]$

$$\left\|\frac{\partial^{s+k}u}{\partial x^s \partial t^k}\right\| \le \varepsilon^{-s/2} C_p \left\{\left\|u_p(x,p(r-1))\right\|^2 + \int_{p(r-1)}^{pr} \left\|u_p(x,\tau)\right\|^2 d\tau,\right.$$

$+\|F\|^2_{L_2(D)}$ $s = 0,1,2\,; k = 0,1$ [8-12].

2. Construction of Difference Schemes

In this subsection, adaptive mesh points are presented. For these points, the mesh generation function that Bakhvalov 1969, mentioned in his paper is used.

Let ω denote the mesh on D, where $\omega = \omega_N \times \omega_\tau$

$$\omega_N = \{x_i = ih_i,\ \ i = 1,2,\dots,N-1;\ h_i = x_i - x_{i-1}\},$$

$$\omega_\tau = \left\{t_j = j\tau,\ \ j = 1,2,\dots,M;\ \tau = \frac{T}{M}\right\},$$

$$\omega_N^+ = \omega_N \cup \{x = 0, l\}, \bar{\omega}_\tau = \omega_\tau \cup \{t = 0\}$$

Let the mesh function v be defined on ω_N. In this study, the notations used are given by Samarskii [15]:

$$v_x = \frac{v_{i+1}-v_i}{\hbar_{i+1}}, v_{\bar{x}} = \frac{v_i - v_{i-1}}{h_i}, v_{\hat{x}} = \frac{v_{i+1}-v_i}{\hbar_i}, v_{\bar{x}\hat{x}} = \frac{v_x - v_{\bar{x}}}{\hbar_i}$$

3. Let the function $g(t)$ be defined on mesh ω_τ. Then the formulas are the following:

$$g_t = \frac{g_{j+1}-g_i}{\tau}, g_t = \frac{g_j - g_{j-1}}{\tau}, g_{\bar{t}t} = \frac{g_{j+1} - 2g_j + g_{j-1}}{\tau^2}$$

Bakhvalov mesh points [6-7] are as the following:

$$x_i = \begin{cases} -\alpha^{-1}\varepsilon \ln\left(1 - (1-\varepsilon)\frac{4i}{N}\right), i = 0,1,\dots,\frac{N}{2}, x_i \in [0,\sigma_1],\ \text{if}\ \sigma_1 < \frac{l}{2}; \\ -\alpha^{-1}\varepsilon \ln\left(1 - \left(1 - e^{-\frac{\alpha l}{4\varepsilon}}\right)\frac{4i}{N}\right), i = 0,1,\dots,\frac{N}{2}, x_i \in [0,\sigma_1],\ if\ \sigma_1 = \frac{l}{2}; \\ \sigma_1 + \left(i - \left(\frac{N}{4}\right)\right) h^{(1)}, i = \frac{N}{2}+1,\dots,\frac{3N}{2}, x_i \in [\sigma_1, l], h^{(1)} = \frac{2(\sigma_2 - \sigma_1)}{N}; \end{cases}$$

where, $\sigma_1 = \min\left\{\frac{l}{2}, -\alpha^{-1}\varepsilon \ln \varepsilon\right\}$.

We write the following difference schemes:

$$\ell y = y^j_{\bar{t},i} - \varepsilon y^j_{\bar{x}x,i} + a_i y^j_{x,i} + b^j_i y^j_i$$
$$= f\left(x_i, t_j, y(x_i, t_j)\right) - c^j y_i^{j-M_0},\ \ i = 1,\dots,N-1; j = 1,\dots,M, \qquad (7)$$

$$y(x_i, t_j) = \psi(x_i, t_j); \quad -M_0 \le j \le 0, \ 0 \le i \le N, \tag{8}$$
$$y(0, t_j) = y(l, t_j), \qquad t_j \in \omega_\tau \tag{9}$$

where, $t_j - r = j\tau - r\frac{T}{M}\frac{M}{T} = j\tau - r\frac{T}{M}\frac{M}{rm} = j\tau - \frac{T}{M}\frac{M}{m} = j\tau - \tau M_0 = \tau(j - M_0) = t_{j-M_0}$, $\frac{M}{m} = M_0$.

4. Error Estimation

Lemma Let be respectively the solutions of the problem (1)-(3) and the solution of the problem (7)-(9) u and y. The following estimation is true:

$$\left|y_i^j - z_i^j\right| \le C(N^{-2} + \tau).$$

Conclusion. A finite difference method based on the mesh with adaptive points are proposed. The method employs interpolating quadrature rules containing integral remainder terms with linear basis functions ensuring a second-order accuracy rate on an adaptive mesh. The proposed method exhibits second-order convergence in the space variable, and first-order in the time variable, regardless of the perturbation parameter. The results demonstrate the effectiveness of the proposed approach in accurately solving singularly perturbed nonlinear delay parabolic problems with boundary value conditions.

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КЕШЕУІЛДЕЙТІН ЖӘНЕ ПЕРИОДТЫ БАСТАПҚЫ-ШЕТКІ ЖАҒДАЙЛАРЫ БАР СИНГУЛЯРЛЫ БҰЗЫЛҒАН СЫЗЫҚТЫҚ ЕМЕС ПАРАБОЛАЛЫҚ ЕСЕПТЕР ҮШІН ЕКІНШІ РЕТТІ СОҢҒЫ АЙЫРМАШЫЛЫҚТАР ӘДІСІ

Аңдатпа. Бұл жұмыста артта қалған және шекаралық жағдайлары бар сызықтық емес параболалық есептерге жүргізілген сандық зерттеу нәтижелері келтірілген. Адаптивті нүктелері бар торға негізделген ақырлы айырмашылықтар әдісі ұсынылады. Әдіс адаптивті торда екінші ретті дәлдікті қамтамасыз ететін сызықтық негіз функциялары бар интегралды қалдық мүшелерден тұратын интерполяциялық квадратуралық ережелерді қолданады. Ұсынылған әдіс кеңістіктік айнымалы бойынша екінші ретті және бұзылу параметріне қарамастан уақыт айнымалысы бойынша бірінші ретті конвергенцияны көрсетеді. Тұрақтылықты талдау келтірілген, ал сандық эксперименттер теориялық тұжырымдарды қолдайды. Нәтижелер артта қалған және шекаралық жағдайлары бар сингулярлы бұзылған сызықтық емес параболалық есептерді дәл шешу үшін ұсынылған тәсілдің тиімділігін көрсетеді.

Тірек сөздер: парабола-кешіктірілген эллиптикалық есептер, асимптотикалық талдау, ақырлы айырмашылық әдісі, біркелкі конвергенция, Бахвалов торы.

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МЕТОД КОНЕЧНЫХ РАЗНОСТЕЙ ВТОРОГО ПОРЯДКА ДЛЯ СИНГУЛЯРНО ВОЗМУЩЕННЫХ НЕЛИНЕЙНЫХ ПАРАБОЛИЧЕСКИХ ЗАДАЧ С ЗАПАЗДЫВАНИЕМ И ПЕРИОДИЧЕСКИМИ НАЧАЛЬНО-КРАЕВЫМИ УСЛОВИЯМИ

Аннотация. В данной работе представлено численное исследование сингулярно возмущенных нелинейных параболических задач с запаздыванием и граничными условиями. Предлагается метод конечных разностей на основе сетки с адаптивными точками. В методе используются интерполирующие квадратурные правила, содержащие интегральные остаточные члены с линейными базисными функциями, обеспечивающие точность второго порядка на адаптивной сетке. Предлагаемый метод демонстрирует сходимость второго порядка по пространственной переменной и первого порядка по временной переменной, независимо от параметра возмущения. Приведен анализ устойчивости, а численные эксперименты подтверждают теоретические выводы. Полученные результаты демонстрируют эффективность предложенного подхода для точного решения сингулярно возмущенных нелинейных параболических задач с запаздыванием и граничными условиями.

Ключевые слова: параболо-эллиптические задачи с запаздыванием, асимптотический анализ, метод конечных разностей, равномерная сходимость, сетка Бахвалова.