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COMPUTATION OF THE BEST STANDARD DEVIATION BY LEJANDRE POLYNOMIALS IN A COMPUTER MATHEMATICS SYSTEM

Abstract. The extensive practical applications of Lejandre polynomials, in conjunction with the development of computer mathematics systems, have engendered a series of prerequisites that are conducive to the expansion of the possibilities for the application of mathematical methods. The contemporary challenge lies in identifying effective calculation and application methodologies for these polynomials within the framework of analytical calculations. The present paper proposes a code for calculating the best standard deviation by Lejandre polynomials with graphical visualisation in Maple.

Keywords: function decomposition, orthogonal polynomials, approximation, series, Rodrigue formula.

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Introduction. The theory of orthogonal polynomials has a vast field of theoretical and practical applications. The distinctive characteristic of these polynomials, which is evident in their notable properties, engenders the potential for their utilization in a myriad of application problems. Orthogonal polynomials are characterized by their ability to interpolate and approximate functions, a property that is crucial in a wide range of applications.

A separate class of orthogonal polynomials is Lejandre polynomials. To determine the $P_n(x)$ Lejandre polynomials of degree n the Rodrigue formula is used [1]:

$$P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}\left(x^2 - 1\right)^n,$$

which leads to explicit expressions of these polynomials. Thus, at $n = \overline{0,4}$ the Lejandre polynomials have the following representation:

$P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$, $P_3(x)=\frac{1}{2}(5x^3-3x)$, $P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$.

Lejandre $P_n(x)$ polynomials have one of the properties that allows us to use them in the best way when approximating functions. Due to the completeness of $P_n(x)$ on the interval [-1,1], it is possible to construct a system of polynomials which will have the smallest deviation from the given function in the sense of RMS approximation.

Materials and methods. The theoretical justification for the construction of the system of Lejandre $P_n(x)$ polynomials for a given function $y = f(x)$ on the segment [-1,1] is the following theorem [2]:

Every twice continuously differentiable function $f(x)$ on the interval [-1,1] has a Lejandre polynomial series expansion which is absolutely and uniformly convergent:

$$f(x)=\sum_{n=0}^{\infty} f_n(x)P_n(x), \tag{1}$$

where $f_n(x)$ is the Fourier coefficients of the $f(x)$ function:

$$f_n(x)=\frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx, \tag{2}$$

where $n = 0,1,\ldots,\ldots$.

Let $PL_n(x)$ be a polynomial of the form (1):

$$PL_n(x)=\sum_{n=0}^{k} C_n(x)P_n(x) \tag{3}$$

Then formula (2) is used to calculate the coefficients $C_n(x)$, where $n = 0,1,\ldots,k$, $P_n(x)$ are Lejandre polynomials and Fourier coefficients:

$$C_n(x)=\frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx. \tag{4}$$

As a measure of the closeness of the function $f(x)$ to $PL_n(x)$ we take the deviation of the RMS approximation σ_n^2 to $y = f(x)$:

$$\sigma_n^2=\int_{-1}^{1} f^2(x)dx-\sum_{n=0}^{k}\frac{2}{2n+1}C_n^2(x). \tag{5}$$

Example. For a function $y = e^x$ on [-5,2], find the best RMS approximation of σ_n^2 [3].

Solution. The segment [-5,2] on which we need to find σ_n^2 does not fulfil the requirement of the theorem: [-1,1]. Therefore, we introduce a variable substitution of the function $y = e^x$. Let the variable substitution have a linear form:

$$x = at + b \tag{6}$$

According to (6), we form a system of equations:

$$\begin{cases} at_1 + b = x_1 \\ at_2 + b = x_2 \end{cases}$$

where $x_1 = -5$, $t_1 = -1$, $x_2 = 2$, $t_2 = 1$ and we find the values of and: b: $a = \frac{7}{2}$, $b = -\frac{3}{2}$.
We have a variable substitution:

$$x = \frac{7}{2}t - \frac{3}{2}, \tag{7}$$

which converts $x \in [-5, 2]$ to $t \in [-1,1]$. The function $y = e^x$ at (7) takes the form:

$$F(t) = e^{\frac{7}{2}t - \frac{3}{2}}. \tag{7}$$

Let us compute $C_n(t)$ by (4), assuming $n = \overline{0,4}$ and using the representation $P_n(t)$ by *(A):*

$$C_0(t) = \frac{2\cdot 0+1}{2}\int_{-1}^{1} e^{\frac{7}{2}t-\frac{3}{2}} \cdot P_0(t)dt = \frac{1}{2}\int_{-1}^{1} e^{\frac{7}{2}t-\frac{3}{2}} \cdot 1 \cdot dt = -\frac{1}{7}e^{-5} + \frac{1}{7}e^{2};$$

$$C_1(t) = \frac{2\cdot 1+1}{2}\int_{-1}^{1} e^{\frac{7}{2}t-\frac{3}{2}} \cdot P_1(t)dt = \frac{3}{2}\int_{-1}^{1} e^{\frac{7}{2}t-\frac{3}{2}} \cdot t \cdot dt = \frac{27}{49}e^{-5} + \frac{15}{49}e^{2};$$

$$C_2(t) = \frac{2\cdot 2+1}{2}\int_{-1}^{1} e^{\frac{7}{2}t-\frac{3}{2}} \cdot P_2(t)dt = \frac{5}{2}\int_{-1}^{1} e^{\frac{7}{2}t-\frac{3}{2}} \cdot \frac{3t^2-1}{2} \cdot dt = \frac{515}{343}e^{-5} + \frac{95}{343}e^{2};$$

$$C_3(t) = \frac{2\cdot 3+1}{2}\int_{-1}^{1} e^{\frac{7}{2}t-\frac{3}{2}} \cdot P_3(t)dt = \frac{5}{2}\int_{-1}^{1} e^{\frac{7}{2}t-\frac{3}{2}} \cdot \frac{5t^3-3t}{2} \cdot dt = \frac{1471}{343}e^{-5} + \frac{55}{343}e^{2};$$

$$C_4(t) = \frac{2\cdot 4+1}{2}\int_{-1}^{1} e^{\frac{7}{2}t-\frac{3}{2}} \cdot P_4(t)dt = \frac{9}{2}\int_{-1}^{1} e^{\frac{7}{2}t-\frac{3}{2}} \cdot \frac{35t^4-30t^2+3}{8} \cdot dt = -\frac{32967}{2401}e^{-5} + \frac{207}{2401}e^{2}.$$

Let's calculate $PL_n(t)$ for $n = \overline{0,4}$ at $t \in [-1,1]$ for (7) using the formula (1):

$$PL_0(t) = C_0(t) \cdot P_0(t) = -\frac{1}{7}e^{-5} + \frac{1}{7}e^{2};$$

$$PL_1(t) = \sum_{n=0}^{1} C_n(t)P_n(t) = C_0(t)\cdot P_0(t) + C_1(t)\cdot P_1(t) = -\frac{1}{7}e^{-5} + \frac{1}{7}e^{2} + \left(\frac{27}{49}e^{-5} + \frac{15}{49}e^{2}\right)\cdot t;$$

$$PL_2(t)=\sum_{n=0}^{2}C_n(t)P_n(t)=\frac{417}{686}e^{-5}+\frac{3}{686}e^{2}+\left(\frac{27}{49}e^{-5}+\frac{15}{49}e^{2}\right)\cdot t+\left(-\frac{1545}{686}e^{-5}+\frac{285}{686}e^{2}\right)\cdot t^{2};$$

$$PL_3(t)=\sum_{n=0}^{3}C_n(t)P_n(t)=\frac{417}{686}e^{-5}+\frac{3}{686}e^{2}+\left(-\frac{4035}{686}e^{-5}+\frac{45}{686}e^{2}\right)\cdot t+\left(-\frac{1545}{686}e^{-5}+\frac{285}{686}e^{2}\right)\cdot t^{2}+$$
$$+\left(\frac{7355}{686}e^{-5}+\frac{275}{686}e^{2}\right)\cdot t^{3};$$

$$PL_4(t)=\sum_{n=0}^{4}C_n(t)P_n(t)=-\frac{87225}{19208}e^{-5}+\frac{705}{19208}e^{2}+\left(-\frac{4035}{686}e^{-5}+\frac{45}{686}e^{2}\right)\cdot t+\left(\frac{472875}{9604}e^{-5}+\frac{885}{9604}e^{2}\right)\cdot t^{2}+$$
$$+\left(\frac{7355}{686}e^{-5}+\frac{275}{686}e^{2}\right)\cdot t^{3}+\left(-\frac{164835}{2744}e^{-5}+2744e^{2}\right)\cdot t^{4}$$

Using variable substitution $t=\frac{2}{7}x+\frac{3}{7}$, we calculate $PL_n(x)$ for $n=\overline{0,4}$ at $x\in[-5,2]$ for $y=e^{x}$:

$$PL_0(x)=-\frac{1}{7}e^{-5}+\frac{1}{7}e^{2};$$
$$PL_1(x)=\frac{1}{7}e^{-5}+\frac{1}{7}e^{2}+\left(\frac{27}{49}e^{-5}+\frac{15}{49}e^{2}\right)\cdot\left(\frac{2}{7}x+\frac{3}{7}\right)=\frac{32}{343}e^{-5}+\frac{52}{49}e^{2}+\left(\frac{54}{343}e^{-5}+\frac{30}{343}e^{2}\right)\cdot x;$$

$$PL_2(x)=\frac{7233}{16807}e^{-5}+\frac{3561}{16807}e^{2}+\left(-\frac{6624}{16807}e^{-5}+\frac{3180}{16807}e^{2}\right)\cdot x+\left(-\frac{3090}{16807}e^{-5}+\frac{570}{16807}e^{2}\right)\cdot x^{2};$$

$$PL_3(x)=-\frac{174432}{117649}e^{-5}+\frac{16512}{117649}e^{2}+\left(-\frac{64020}{117649}e^{-5}+\frac{21600}{117649}e^{2}\right)\cdot x+\left(\frac{110760}{117649}e^{-5}+\frac{8940}{117649}e^{2}\right)\cdot x^{2}+;$$
$$+\left(\frac{29420}{117649}e^{-5}+\frac{1100}{117649}e^{2}\right)\cdot x^{3}$$

$$PL_4(x)=-\frac{658095}{823543}e^{-5}+\frac{10378}{823543}e^{2}+\left(\frac{5485920}{823543}e^{-5}+\frac{113940}{823543}e^{2}\right)\cdot x+\left(-\frac{213690}{823543}e^{-5}+\frac{68790}{823543}e^{2}\right)\cdot x^{2}+$$
$$\left(-\frac{1772080}{823543}e^{-5}+\frac{20120}{823543}e^{2}\right)\cdot x^{3}+\left(-\frac{329670}{823543}e^{-5}+\frac{22070}{823543}e^{2}\right)\cdot x^{4}$$

We use formula (5) to calculate the RMS approximation $PL_n(x)$:

$$\sigma_0^2=\int_{-1}^{1}f^2(t)dt-2C_0^2(t)\approx 5{,}575295713;$$

$$\sigma_1^2=\int_{-1}^{1}f^2(t)dt-2C_0^2(t)-\frac{2}{3}C_1^2(t)\approx 2{,}153125955$$

$$\sigma_2^2=\int_{-1}^{1}f^2(t)dt-\sum_{n=0}^{2}\frac{2}{2n+1}C_n^2(t)\approx 0{,}49431813;$$

$$\sigma_3^2 = \int_{-1}^{1} f^2(t)dt - \sum_{n=0}^{3} \frac{2}{2n+1} C_n^2(t) \approx 0{,}0734340882;$$

$$\sigma_4^2 = \int_{-1}^{1} f^2(t)dt - \sum_{n=0}^{4} \frac{2}{2n+1} C_n^2(t) \approx 0{,}00754351905.$$

Comparing the values σ_n^2 with $n = \overline{0,4}$, we conclude that the smallest deviation is σ_4^2.

To reduce the labour intensity of computational processes, we will solve the example in the computer mathematics system Maple.

For calculations with orthogonal polynomials in Maple a special package *orthopoly*. is designed [4].

```
restart; with(orthopoly);
[G, H, L, P, T, U]
```

The package includes 6 functions. Each letter of the functions is the first letter present in the name of the orthogonal polynomial. For $P_n(x)$, the letter P is used. The command syntax for calling $P_n(x)$is written as: $P(n,x)$, where *n* is the degree , and *x* is the variable of the polynomial [4]. For example:

```
P0 := P(0, x); P1x := P(1, x); P2t := P(2, t);  P0 := 1
P1x := x
P2t := -1/2 + 3/2 t^2
```

Let's solve the example in Maple. Define a function and the segment on which it is considered: $f := \exp(x); x1 := -5; x2 := 2;$

```
f := e^x
x1 := -5
```

Then, we introduce the boundary points of the segment [-1,1]:

```
t1 := -1; t2 := 1;
t1 := -1
t2 := 1
```

We make a system of equations to find the expression of the new variable and solve it:

```
sys1 := {a·t1 + b = x1, a·t2 + b = x2};
rsys1 := solve(sys1, {a, b});
X := rhs(rsys1[1])·t + rhs(rsys1[2]);
sys1 := {-a + b = -5, a + b = 2}
rsys1 := {a = 7/2, b = -3/2}
```

$$X := \frac{7}{2}t - \frac{3}{2}$$

Since the solution requires that we go back to the variable X, we prescribe actions for this:

$$eq1 := rhs(rsys1[1])\cdot t + rhs(rsys1[2]) = X\_1; req1 := solve(eq1, t);$$

$$eq1 := \frac{7}{2}t - \frac{3}{2} = X\_1$$
$$req1 := \frac{2}{7}X\_1 + \frac{3}{7}$$

Let's compose a function with a new variable:

$$F := subs(x = X, f);$$
$$F := e^{\frac{7}{2}t - \frac{3}{2}}$$

To calculate the coefficients $C_n(t)$ let's make a cycle with a construction, taking as a $n = 4$ [5]:

$$\textbf{for } i \textbf{ from } 0 \textbf{ to } 4 \textbf{ do } C[i] := \frac{(2\cdot i + 1)}{2}\cdot int(F\cdot P(i, t), t = -1..1); \textbf{ od};$$

$$C_0 := -\frac{1}{7}e^{-5} + \frac{1}{7}e^2$$
$$C_1 := \frac{27}{49}e^{-5} + \frac{15}{49}e^2$$
$$C_2 := -\frac{515}{343}e^{-5} + \frac{95}{343}e^2$$
$$C_3 := \frac{1471}{343}e^{-5} + \frac{55}{343}e^2$$
$$C_4 := -\frac{32967}{2401}e^{-5} + \frac{207}{2401}e^2$$

If necessary, by increasing *n* the following coefficients are found $C_n(t)$.

Let's calculate $PL_n(t)$ – approximate Lejandre polynomials [4]:

$$Pl0t := sum(C[k]\cdot P(k, t), k = 0..0);$$
$$Pl1t := sum(C[k]\cdot P(k, t), k = 0..1);$$
$$Pl2t := sum(C[k]\cdot P(k, t), k = 0..2);$$
$$Pl3t := sum(C[k]\cdot P(k, t), k = 0..3);$$
$$Pl4t := sum(C[k]\cdot P(k, t), k = 0..4);$$

$$Pl0t := -\frac{1}{7}e^{-5} + \frac{1}{7}e^2$$
$$Pl1t := -\frac{1}{7}e^{-5} + \frac{1}{7}e^2 + \left(\frac{27}{49}e^{-5} + \frac{15}{49}e^2\right)t$$
$$Pl2t := -\frac{1}{7}e^{-5} + \frac{1}{7}e^2 + \left(\frac{27}{49}e^{-5} + \frac{15}{49}e^2\right)t + \left(-\frac{515}{343}e^{-5} + \frac{95}{343}e^2\right)\left(-\frac{1}{2} + \frac{3}{2}t^2\right)$$

$$Pl3t := -\frac{1}{7}e^{-5} + \frac{1}{7}e^{2} + \left(\frac{27}{49}e^{-5} + \frac{15}{49}e^{2}\right)t + \left(-\frac{515}{343}e^{-5} + \frac{95}{343}e^{2}\right)\left(-\frac{1}{2} + \frac{3}{2}t^{2}\right) + \left(\frac{1471}{343}e^{-5} + \frac{55}{343}e^{2}\right)\left(\frac{5}{2}t^{3} - \frac{3}{2}t\right)$$

$$Pl4t := -\frac{1}{7}e^{-5} + \frac{1}{7}e^{2} + \left(\frac{27}{49}e^{-5} + \frac{15}{49}e^{2}\right)t + \left(-\frac{515}{343}e^{-5} + \frac{95}{343}e^{2}\right)\left(-\frac{1}{2} + \frac{3}{2}t^{2}\right) + \left(\frac{1471}{343}e^{-5} + \frac{55}{343}e^{2}\right)\left(\frac{5}{2}t^{3} - \frac{3}{2}t\right) + \left(-\frac{32967}{2401}e^{-5} + \frac{207}{2401}e^{2}\right)\left(\frac{3}{8} + \frac{35}{8}t^{4} - \frac{15}{4}t^{2}\right)$$

We proceed to the calculation of $PL_n(x)$ – approximate Lejandre polynomials with the variable *x*, but first write through the variable X_1 [6]:

```
Pl0_x := subs(t = req1, Pl0t);
Pl1_x := subs(t = req1, Pl1t);
Pl2_x := subs(t = req1, Pl2t);
Pl3_x := subs(t = req1, Pl3t);
Pl4_x := subs(t = req1, Pl3t);
```

$$Pl0\_x := -\frac{1}{7}e^{-5} + \frac{1}{7}e^{2}$$

$$Pl1\_x := -\frac{1}{7}e^{-5} + \frac{1}{7}e^{2} + \left(\frac{27}{49}e^{-5} + \frac{15}{49}e^{2}\right)\left(\frac{2}{7}X\_1 + \frac{3}{7}\right)$$

$$Pl2\_x := -\frac{1}{7}e^{-5} + \frac{1}{7}e^{2} + \left(\frac{27}{49}e^{-5} + \frac{15}{49}e^{2}\right)\left(\frac{2}{7}X\_1 + \frac{3}{7}\right) + \left(-\frac{515}{343}e^{-5} + \frac{95}{343}e^{2}\right)\left(-\frac{1}{2} + \frac{3}{2}\left(\frac{2}{7}X\_1 + \frac{3}{7}\right)^{2}\right)$$

$$Pl3\_x := -\frac{1}{7}e^{-5} + \frac{1}{7}e^{2} + \left(\frac{27}{49}e^{-5} + \frac{15}{49}e^{2}\right)\left(\frac{2}{7}X\_1 + \frac{3}{7}\right) + \left(-\frac{515}{343}e^{-5} + \frac{95}{343}e^{2}\right)\left(-\frac{1}{2} + \frac{3}{2}\left(\frac{2}{7}X\_1 + \frac{3}{7}\right)^{2}\right) + \left(\frac{1471}{343}e^{-5} + \frac{55}{343}e^{2}\right)\left(\frac{5}{2}\left(\frac{2}{7}X\_1 + \frac{3}{7}\right)^{3} - \frac{3}{7}X\_1 - \frac{9}{14}\right)$$

$$Pl4\_x := -\frac{1}{7}e^{-5} + \frac{1}{7}e^{2} + \left(\frac{27}{49}e^{-5} + \frac{15}{49}e^{2}\right)\left(\frac{2}{7}X\_1 + \frac{3}{7}\right) + \left(-\frac{515}{343}e^{-5} + \frac{95}{343}e^{2}\right)\left(-\frac{1}{2} + \frac{3}{2}\left(\frac{2}{7}X\_1 + \frac{3}{7}\right)^{2}\right) + \left(\frac{1471}{343}e^{-5} + \frac{55}{343}e^{2}\right)\left(\frac{5}{2}\left(\frac{2}{7}X\_1 + \frac{3}{7}\right)^{3} - \frac{3}{7}X\_1 - \frac{9}{14}\right)$$

Now, let’s go to the original variable *x* [4]:

```
Pl0x := subs(X_1 = x, Pl0_x);
Pl1x := subs(X_1 = x, Pl1_x);
Pl2x := subs(X_1 = x, Pl2_x);
Pl3x := subs(X_1 = x, Pl3_x);
Pl4x := subs(X_1 = x, Pl4_x);
```

$$Pl0x := -\frac{1}{7}e^{-5} + \frac{1}{7}e^{2}$$

$$Pl1x := -\frac{1}{7}e^{-5} + \frac{1}{7}e^{2} + \left(\frac{27}{49}e^{-5} + \frac{15}{49}e^{2}\right)\left(\frac{2}{7}x + \frac{3}{7}\right)$$

$$Pl2x := -\frac{1}{7}\,e^{-5} + \frac{1}{7}\,e^{2} + \left(\frac{27}{49}\,e^{-5} + \frac{15}{49}\,e^{2}\right)\left(\frac{2}{7}\,x + \frac{3}{7}\right) + \left(-\frac{515}{343}\,e^{-5} + \frac{95}{343}\,e^{2}\right)\left(-\frac{1}{2} + \frac{3}{2}\left(\frac{2}{7}\,x + \frac{3}{7}\right)^{2}\right)$$

$$Pl3x := -\frac{1}{7}\,e^{-5} + \frac{1}{7}\,e^{2} + \left(\frac{27}{49}\,e^{-5} + \frac{15}{49}\,e^{2}\right)\left(\frac{2}{7}\,x + \frac{3}{7}\right) + \left(-\frac{515}{343}\,e^{-5} + \frac{95}{343}\,e^{2}\right)\left(-\frac{1}{2} + \frac{3}{2}\left(\frac{2}{7}\,x + \frac{3}{7}\right)^{2}\right) + \left(\frac{1471}{343}\,e^{-5} + \frac{55}{343}\,e^{2}\right)\left(\frac{5}{2}\left(\frac{2}{7}\,x + \frac{3}{7}\right)^{3} - \frac{3}{7}\,x - \frac{9}{14}\right)$$

$$Pl4x := -\frac{1}{7}\,e^{-5} + \frac{1}{7}\,e^{2} + \left(\frac{27}{49}\,e^{-5} + \frac{15}{49}\,e^{2}\right)\left(\frac{2}{7}\,x + \frac{3}{7}\right) + \left(-\frac{515}{343}\,e^{-5} + \frac{95}{343}\,e^{2}\right)\left(-\frac{1}{2} + \frac{3}{2}\left(\frac{2}{7}\,x + \frac{3}{7}\right)^{2}\right) + \left(\frac{1471}{343}\,e^{-5} + \frac{55}{343}\,e^{2}\right)\left(\frac{5}{2}\left(\frac{2}{7}\,x + \frac{3}{7}\right)^{3} - \frac{3}{7}\,x - \frac{9}{14}\right)$$

Let's calculate the deviation from the original function [5]:

$$crkv\_0 := int\left(F^2, t=-1..1\right) - sum\left(2\cdot\frac{C[m]^2}{(2\cdot m+1)}, m=0..0\right);$$

$$crkv\_0 := -\frac{1}{7}\,e^{-10} + \frac{1}{7}\,e^{4} - 2\left(-\frac{1}{7}\,e^{-5} + \frac{1}{7}\,e^{2}\right)^{2}$$

For the numerical result of the analytical expression, we use the command *evalf*:

$$crkv0 := evalf\left(int\left(F^2, t=-1..1\right) - sum\left(2\cdot\frac{C[m]^2}{(2\cdot m+1)}, m=0..0\right)\right);$$

$$crkv0 := 5.575295713$$

We proceed as follows and calculate the numerical result of the RMS approximation value:

$$crkv\_1 := int\left(F^2, t=-1..1\right) - sum\left(2\cdot\frac{C[m]^2}{(2\cdot m+1)}, m=0..1\right);$$

$$crkv1 := evalf\left(int\left(F^2, t=-1..1\right) - sum\left(2\cdot\frac{C[m]^2}{(2\cdot m+1)}, m=0..1\right)\right);$$

$$crkv\_2 := int\left(F^2, t=-1..1\right) - sum\left(2\cdot\frac{C[m]^2}{(2\cdot m+1)}, m=0..2\right);$$

$$crkv2 := evalf\left(int\left(F^2, t=-1..1\right) - sum\left(2\cdot\frac{C[m]^2}{(2\cdot m+1)}, m=0..2\right)\right);$$

$$crkv\_3 := int\left(F^2, t=-1..1\right) - sum\left(2\cdot\frac{C[m]^2}{(2\cdot m+1)}, m=0..3\right);$$

$$crkv3 := evalf\left(int\left(F^2, t=-1..1\right) - sum\left(2\cdot\frac{C[m]^2}{(2\cdot m+1)}, m=0..3\right)\right);$$

$$crkv\_4 := int\left(F^2, t=-1..1\right) - sum\left(2\cdot\frac{C[m]^2}{(2\cdot m+1)}, m=0..4\right);$$

$$crkv4 := evalf\left(int\left(F^2, t=-1..1\right) - sum\left(2\cdot\frac{C[m]^2}{(2\cdot m+1)}, m=0..4\right)\right);$$

$$crkv\_1 := -\frac{1}{7}e^{-10} + \frac{1}{7}e^{4} - 2\left(-\frac{1}{7}e^{-5} + \frac{1}{7}e^{2}\right)^2 - \frac{2}{3}\left(\frac{27}{49}e^{-5} + \frac{15}{49}e^{2}\right)^2$$

$crkv1 := 2.153125955$

$$crkv\_2 := -\frac{1}{7}e^{-10} + \frac{1}{7}e^{4} - 2\left(-\frac{1}{7}e^{-5} + \frac{1}{7}e^{2}\right)^2 - \frac{2}{3}\left(\frac{27}{49}e^{-5} + \frac{15}{49}e^{2}\right)^2 - \frac{2}{5}\left(-\frac{515}{343}e^{-5} + \frac{95}{343}e^{2}\right)^2$$

$$crkv\_3 := -\frac{1}{7}e^{-10} + \frac{1}{7}e^{4} - 2\left(-\frac{1}{7}e^{-5} + \frac{1}{7}e^{2}\right)^2 - \frac{2}{3}\left(\frac{27}{49}e^{-5} + \frac{15}{49}e^{2}\right)^2 - \frac{2}{5}\left(-\frac{515}{343}e^{-5} + \frac{95}{343}e^{2}\right)^2 - \frac{2}{7}\left(\frac{1471}{343}e^{-5} + \frac{55}{343}e^{2}\right)^2$$

$crkv3 := 0.0734340882$

$$crkv\_4 := -\frac{1}{7}e^{-10} + \frac{1}{7}e^{4} - 2\left(-\frac{1}{7}e^{-5} + \frac{1}{7}e^{2}\right)^2 - \frac{2}{3}\left(\frac{27}{49}e^{-5} + \frac{15}{49}e^{2}\right)^2 - \frac{2}{5}\left(-\frac{515}{343}e^{-5} + \frac{95}{343}e^{2}\right)^2 - \frac{2}{7}\left(\frac{1471}{343}e^{-5} + \frac{55}{343}e^{2}\right)^2 - \frac{2}{9}\left(-\frac{32967}{2401}e^{-5} + \frac{207}{2401}e^{2}\right)^2$$

$crkv4 := 0.00754351905$

As is evident, at the outset of the computational process, the initial data – the function and the segment – were entered, and then the segment by theorem was entered. It is important to note that all subsequent calculations are performed automatically. The ensuing calculations will be accompanied by graphical visualization.

It is acknowledged that functions can be approximated by polynomials with uniform approximation, as stipulated by the Weierstrass approximation theorem. To illustrate this, consider a polynomial that is the partial sum of the Maclorean series, on the condition that the function on a specific segment is decomposed into a uniformly convergent series. The Taylor series can also be used. For the purpose of comparison with our solution, let us consider the approximate solution by Taylor series of the original function $y = e^x$:

$T\_2 := taylor(\exp(x), x = 0, 2);$
$T\_2 := 1 + x + O(x^2)$

Let's discard the remainder of the series and write the series as a polynomial:

$T2 := convert(T\_2, polynom);\ \ T2 := 1 + x$

We similarly prescribe the steps for the Taylor series of the same function $y = e^x$ by increasing the number of terms of the series:

$T\_3 := taylor(\exp(x), x = 0, 3); T3 := convert(T\_3, polynom);$
$T\_4 := taylor(\exp(x), x = 0, 4); T4 := convert(T\_4, polynom);$

$$T\_3 := 1 + x + \frac{1}{2}x^2 + O(x^3)$$

$$T3 := 1 + x + \frac{1}{2}x^2$$
$$T\_4 := 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)$$
$$T4 := 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

For visual representation we plot the graphs of the function $y = e^x$, approximated solutions by Lejandre polynomials and Taylor series (Fiqures 1-3) [5]:

```
plot([exp(x), Pl2x, T2], x = -5 .. 2, color = [red, green, blue], linestyle = [solid, dashdot, dash],
    thickness = 3, legend = ["_exp(x)_", "_Многочлены_Лежандра_n=2"_,
    "Ряд_Тейлора_n=2_"]);
```

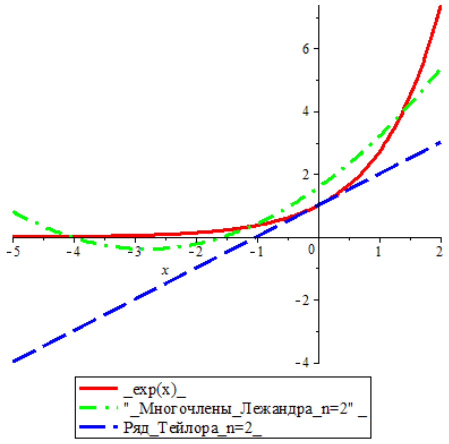

Fig. 1. Graph of building an approximation of the function $y = e^x$. Lejandre polynomials $PL_2(x)$, Taylor series at $n = 2$.

```
plot([exp(x), Pl3x, T3], x = -5 .. 2, color = [red, green, blue], linestyle = [solid, dashdot, dash],
    thickness = 3, legend = ["_exp(x)_", "_Многочлены_Лежандра_n=3"_,
    "Ряд_Тейлора_n=3_"]);
```

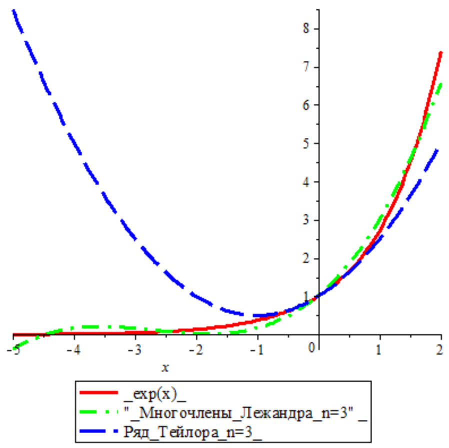

Fig. 2. Graph of building an approximation of the function $y = e^x$. Lejandre polynomials $PL_3(x)$, Taylor series at $n = 3$.

```
plot([exp(x), Pl4x, T4], x = -5 .. 2, color = [red, green, blue], linestyle = [solid, dashdot, dash],
    thickness = 3, legend = ["_exp(x)_", "_Многочлены_Лежандра_n=4"_,
   "Ряд_Тейлора_n=4_"]);
```

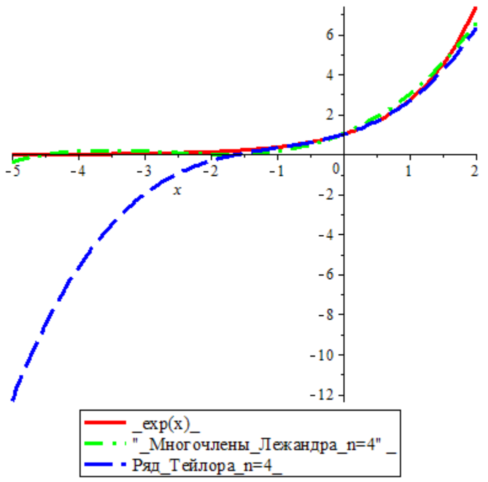

Fig. 3. Graph of building an approximation of the function $y = e^x$. Lejandre polynomials $PL_4(x)$, Taylor series at $n = 4$.

Comparing the graphs of approximation of the function $y = e^x$ by approximate Lejandre polynomials and Taylor series, it is clearly seen that already at $n = 3$ the greatest accuracy of the function graph is achieved by polynomials $PL_n(x)$. While the approximation of the function by the Taylor series is only close to the function in the neighbourhood of *0*.

Research results and discussion. The resolution of a multitude of practical exemplifications has demonstrated that in linear normed spaces, the optimal approximation is attained by approximate Lejandre polynomials $PL_n(x)$. However, it should be noted that the solution of these examples is accompanied by complicated calculations. The labour intensity of the computational process is primarily related to the increase of the degree $P_n(x)$, which directly affects the reduction of the magnitude of the approximate solution.

A study conducted in the Maple computer mathematics system, employing visual representation of function approximation, has demonstrated that the approximate solution by Lejandre polynomials deviates from the exact solution to the smallest extent already at *n = 3*. Furthermore, the approximation of the function by Lejandre polynomials is more proximate to the exact graph of the function than the approximation of the function by Taylor series.

Discussion of scientific results. The practical implementation of the system of computer mathematics has enabled the development of a code for finding the best standard deviation by Lejandre polynomials, with graphical visualization in the Maple program. The developed code has been shown to reduce the time cost of the calculation process and its labour intensity.

Conclusion. The development of modern computer mathematics and the practical applications of problem-solving are creating the conditions for expanding the possibilities of application of mathematical methods. The efficacy of combined methods implemented in the system of analytical calculations, as demonstrated by studies, is a testament to the value of such approaches.

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КОМПЬЮТЕРЛІК МАТЕМАТИКА ЖҮЙЕСІНДЕГІ ЛЕЖАНДР КӨПМҮШЕЛІГІНІҢ ЕҢ ЖАҚСЫ ОРТАША КВАДРАТТЫ АУЫТҚУЫН ЕСЕПТЕУ

Аңдатпа. Лежандр көпмүшелігінің ауқымды тәжірибелік қосымшалары мен компьютерлік математика жүйелерін дамыту математикалық әдістерді пайдалану мүмкіндіктерін кеңейту үшін алғашқы қадам болып табылады. Талдамалық есептеу жүйесінде осы көпмүшеліктерді есептеудің және оларды қолданудың тиімді әдістерін табу қазіргі заманауи мәселелердің бірі болып табылады. Мақалада Maple жүйесінде графикалық визуализациясы бар Лежандр көпмүшелілігінің ең жақсы орташа квадраттық ауытқуды есептеу коды әзірленген.

Тірек сөздер: функцияны жіктеу, ортогональды көпмүшеліктер, жуықтау, қатар, Родрига формуласы.

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ВЫЧИСЛЕНИЕ НАИЛУЧШЕГО СРЕДНЕКВАДРАТИЧНОГО ОТКЛОНЕНИЯ ПОЛИНОМАМИ ЛЕЖАНДРА В СИСТЕМЕ КОМПЬЮТЕРНОЙ МАТЕМАТИКИ

Аннотация. Обширные практические приложения полиномов Лежандра и развитие систем компьютерной математики создают предпосылки для расширения возможностей применения математических методов. Современным вопросом является нахождение эффективных методов вычисления и приложения данных полиномов в системе аналитических вычислений. В статье разрабатывается код вычисления наилучшего среднеквадратичного отклонения полиномами Лежандра с графической визуализацией в системе Maple.

Ключевые слова: разложение функций, ортогональные полиномы, аппроксимация, ряд, формула Родрига.